Investment valuation. Mathematical challenges and solutions

Science of Inexact Mathematics

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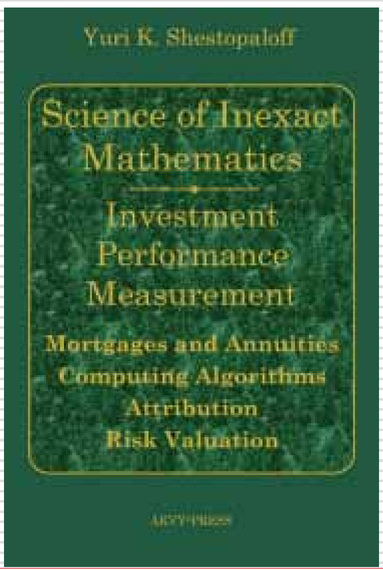
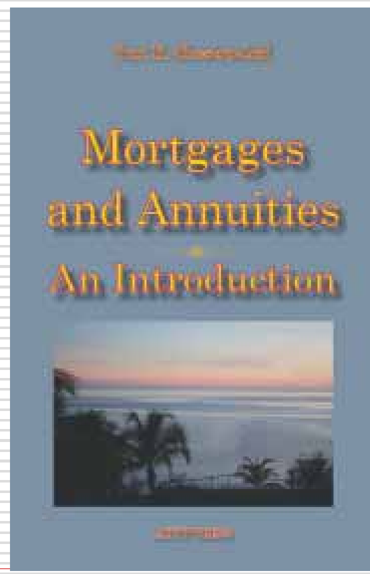
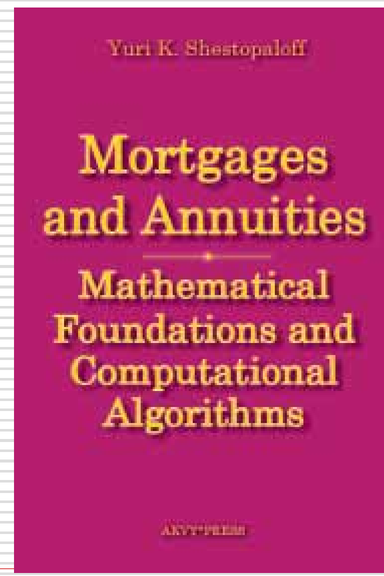
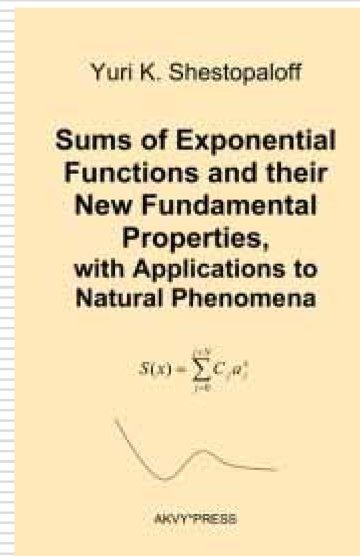
Referred books

- Science of Inexact Mathematics
- Mortgages and Annuities: an Introduction
- Mortgages and Annuities: Mathematical Foundations and Computational Algorithms







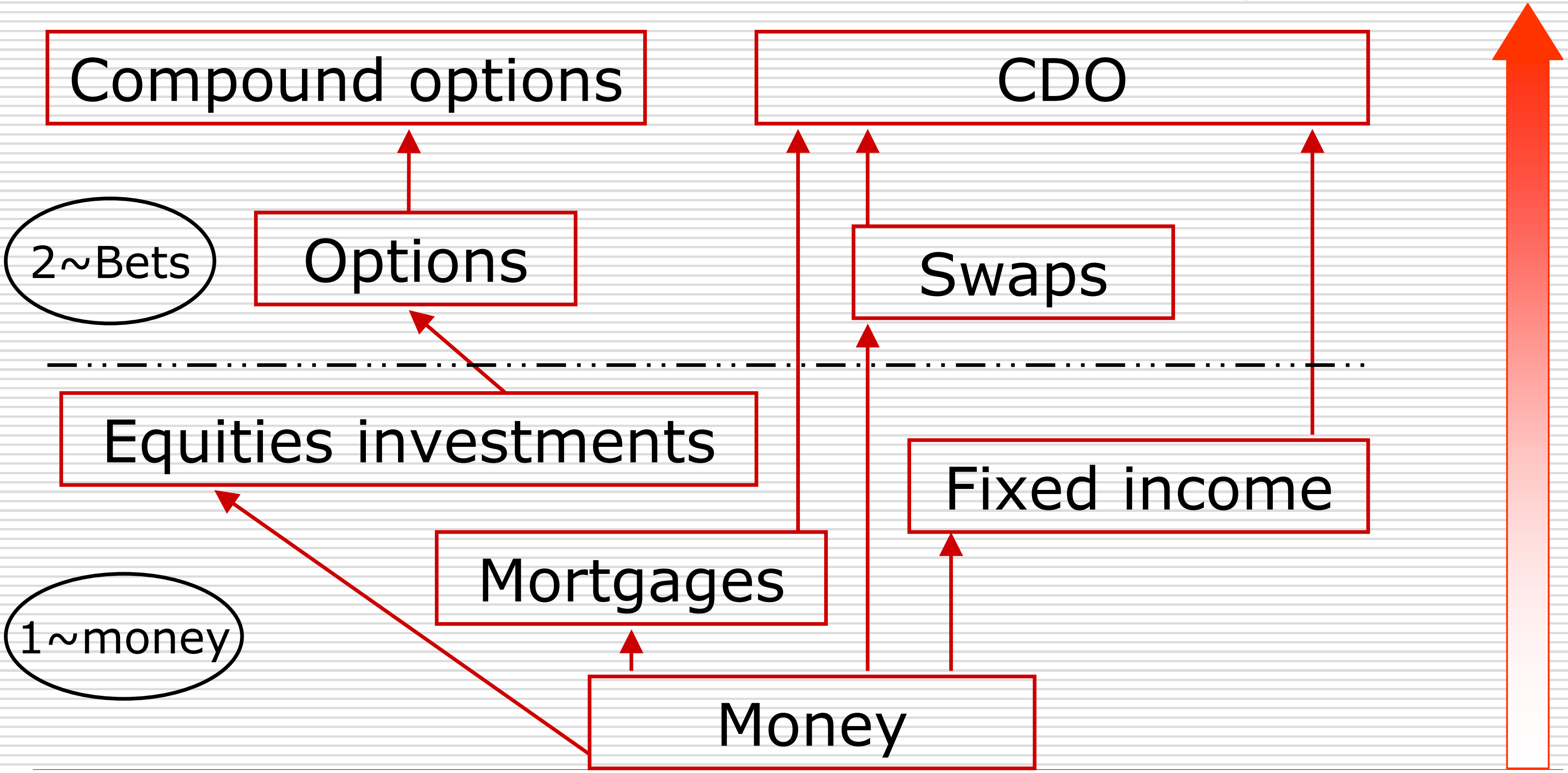


Brief Content

- Diversity of financial instruments
- Compounding and non-compounding contexts
- Linking of rates of return
- Investment performance measurement. IRR equation (rate of return, yield to maturity, solutions' ambiguity, etc.)
- Mortgages, annuities, and bonds
- Performance attribution
- Computational algorithms
- Risk valuation and ratings

Diversity of financial instruments. "Money" instruments and derivatives

- CDO – collateralized debt obligation



More red indicates a higher level of abstraction

Interest rate calculations for different period lengths

Nominal annual interest rate (1) → monthly interest rate → *effective* annual interest rate (2). That is (12% → 1.0% monthly → 12.68%)

(1) ≠ (2)

- Introduction of compounding and non-compounding contexts solves this problem. However, this will require changing some industry practices.

Geometric Linking Theorem

Suppose we have sequential investment periods composing a total period; rates of return for each period are known. Then, the geometric linking operation produces a valid rate of return for the total period **if and only if** two conditions are fulfilled. The first is that there are no cash transactions within the periods. Secondly, the ending market value of each previous period is equal to the beginning market value of the next period.

New methods for linking of rates of return both across assets and periods

	Period 1	Period 2	Period 3	Total ROR Using SL
Asset 1	1.77865612	0.22468793	0.0767590	2.4141176
Asset 2	-0.09836065	-0.2833675	0.0272108	-0.3270440
Asset 3	0.35294117	0.08759124	0.5703422	1.3469387
Total Portfolio	0.62730627	0.03238866	0.1425091	0.8281631
Accumulated return for the total portfolio	0.62730627 Period 1	0.625210084 Period 1 + Period 2	0.8281631 Period 1 + Period 2 + Period 3	

Investment performance measurement

- IRR equation versus other methods:

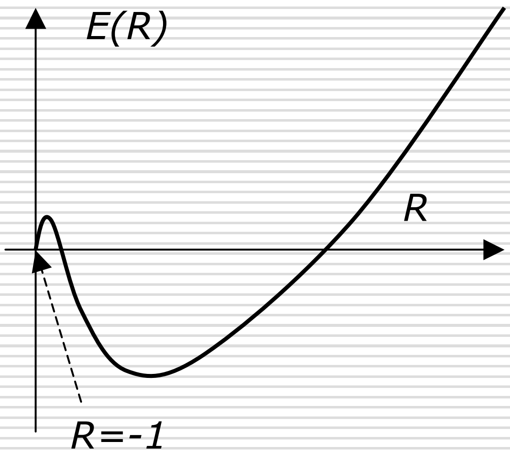
  From the definition of interest rate (rate of return), we can derive *only* the IRR equation. On the other hand, all other methods can be derived from the IRR equation as its *approximations.*
- Introduction of *compounding* and *non-compounding contexts* will make evaluation of investments more objective, although some degree of uncertainty will remain.
- The computed rate of return is in general non-exact, because we *always* solve an *inverse* problem.

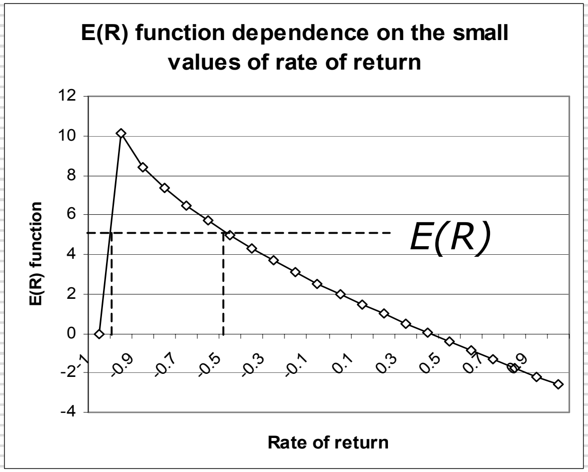
$$E(R) = \sum_{j=0}^{N} C_j (1+R)^{T_j}$$

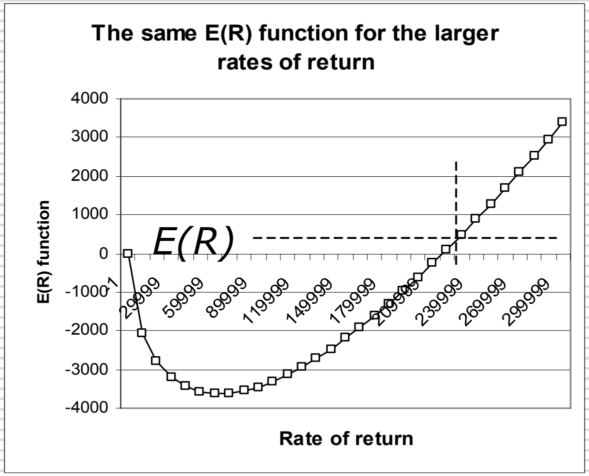
IRR equation. The problem of multiple solutions with respect to rate of return

- Polynomial equations usually have multiple solutions
- Solutions of the IRR equation have some restrictions on them (for example, $R \geq -1$)
- When the theorem about properties of sums of exponential functions is applied to the IRR equation, it turns out that this equation can have a maximum of three solutions

IRR equation. The problem of multiple solutions with respect to rate of return







$$E(R) = \sum_{j=0}^{N} C_j (1+R)^{T_j}$$

C_j={1.0, -3.0, -4.0, -3.0, 11.0}
T_j={0.9, 0.8, 0.6, 0.5, 0.001}

IRR equation. Summary of results

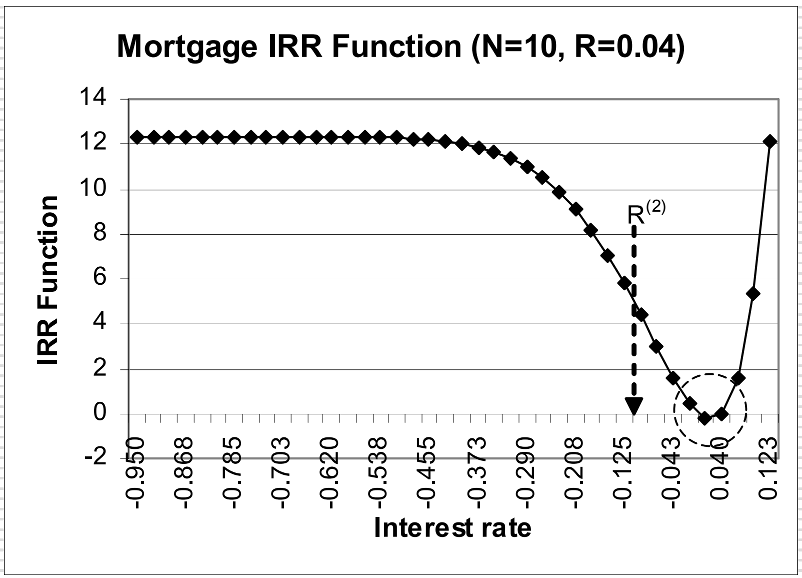
- Although we still have multiple solutions, we now know that the IRR equation can have at most three roots.
- How do we select the correct solution? We pick the middle solution when there are three solutions, and the left solution when there are two solutions.
- So, given the general nature of the IRR equation and the fact that all other methods are derived from it, this property of IRR equation makes its application unambiguous and practical.
- Given the importance of this result for practical applications, some additional verification procedures are very desirable, as well as attempts to find examples of the IRR equation with more than three roots. (Maybe this can be advertised as a challenging problem for mathematicians?)

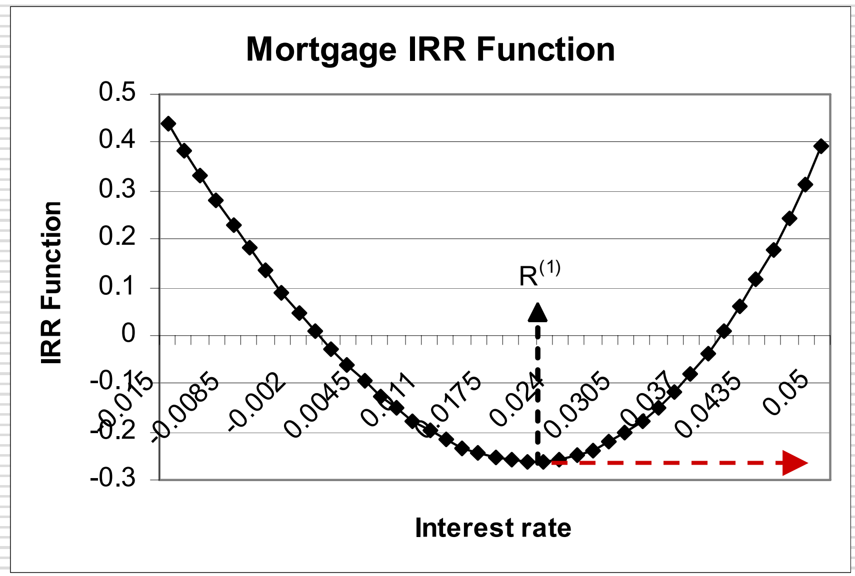
Mortgages and annuities

- Mathematically similar to bonds
- Beneficial features of the mortgage IRR function allow the development of faster computational algorithms. These features also allow us to locate the region of rates of return where the correct solution of the IRR equation is located and is unique
- Available efficient algorithms for rate of return computations are also well-suited for implementation in financial calculators

Mortgage IRR function

$$F(R) = P(1+R)^{n+1} - (P+C)(1+R)^n + C$$





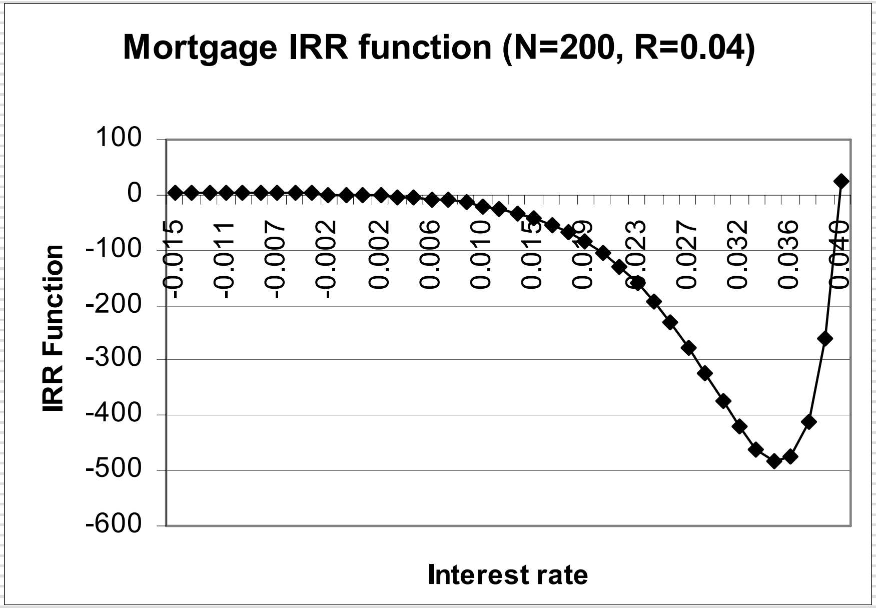
Domain of unique solution

Mortgage IRR function

The function's extrema and inflection points do not depend on the interest rate (rate of return)

$$R_0^{(1)} = \frac{nC - P}{(n+1)P}$$

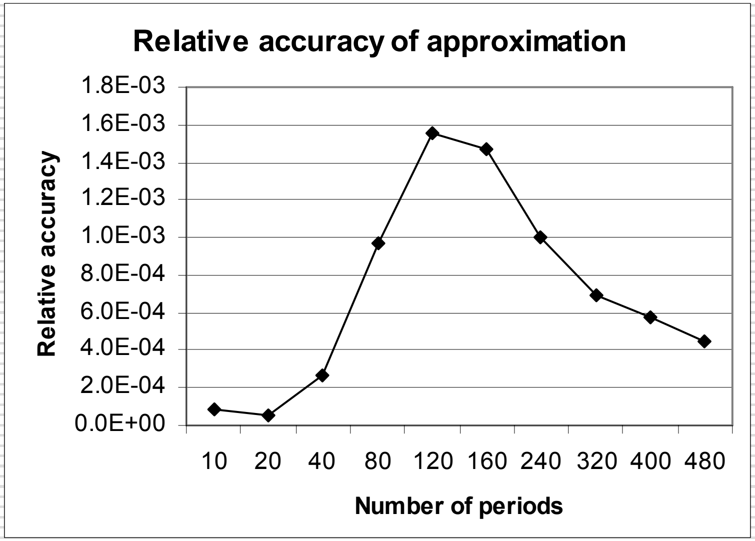
$$R_0^{(2)} = \frac{C(n-1) - 2P}{(n+1)P}$$



Calculation of interest rate

Computational efficiency of quadratic and Newton-Raphson iterations (1-3). Shown is the time required to perform 1,000,000 interest rate calculations, in sec.

Quadratic iteration	Newton-Raphson
0.72 – 0.91	1.06 – 1.37



Computational algorithms: requirements

- Reliability
- Accuracy
- Computational performance
- Efficient software implementation (at the system and code level)
- New groups of algorithms (e.g. for financial calculators, financial systems)

Performance attribution

- Interaction effect conundrum: solved for equities. New methods are more objective and do not have an interaction term. New objective criteria for the development of arithmetic and geometric attribution methods have been introduced
- Example of an arithmetic attribution method. It was derived using two *independent* approaches: through algebraic transformations, and through eigenvalues of a matrix composed of values derived from the definitions of industry selection and stock selection

$$I_s = \frac{(r+R)}{2}(v-W) \qquad S_s = (\frac{v+W}{2})(r-R)$$

Criteria for development of objective attribution methods

- A correct comparison method should be *symmetric*. The result of a comparison should not depend on the direction of the comparison.
- A valid comparison method should allow us to swap the entities we are comparing, and the results of our comparison should be *conjugate*.
- A correct symmetric comparison method excludes noise factors, and *uses only meaningful parameters*.
- Functional symmetry of a comparison method assumes that its mathematical description is symmetrical.
- *Symmetrical data sets* should produce symmetrical attribution parameters.

Ratio validation criterion

The ratios of the same attribution parameters for the same input data should be the equal across different attribution methods

Asset level attribution parameters
Geometric (G) versus arithmetic (A)

$$\frac{I_j^G}{S_j^G} = \frac{I_j^A}{S_j^A}$$

Total attribution parameters
Geometric (G) versus arithmetic (A)

$$\frac{I_T^G}{S_T^G} = \frac{I_T^A}{S_T^A}$$

Examples of applying existing and new methods to the same data

Sample portfolio

Group No.	*R*	*r*	*W*	*v*
1	0.4	0.1	0.2	0.3
2	0.3	0.1	0.3	0.25
3	0.15	0.45	0.35	0.25
4	0.2	0.3	0.15	0.2
Total	**0.2525**	**0.2275**	**1.0**	**1.0**

Present model (BHB)

Group No.	Industry Selection	Stock Sel.	Interac-tion
1	0.04	-0.06	-0.03
2	-0.015	-0.06	0.01
3	-0.015	0.105	-0.03
4	0.01	0.015	0.005
Total	**0.02**	**0.0**	**-0.045**

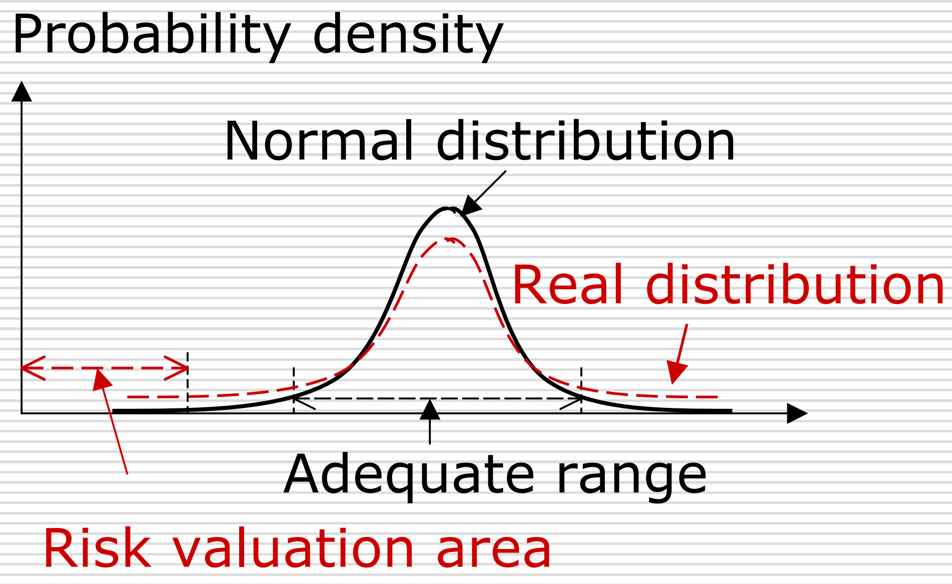
New model (SAA)

Group No.	Industry Sel	Stock Sel.	Interaction
1	0.025	-0.075	0
2	-0.01	-0.055	0
3	-0.03	0.09	0
4	0.0125	0.0175	0
Total	**-0.0025**	**-0.0225**	

Risk valuation and ratings

- Statistical characterizations are often based on the price movements during a certain period of time, which *always* have certain *unique* features
- Normal distribution. Widely used in risk valuation and ratings, but its adequacy is questionable for unlikely events
- Unreliable estimation of rare event probabilities (underestimation)
- For AA3 rating, the default probability is ***0.0003***

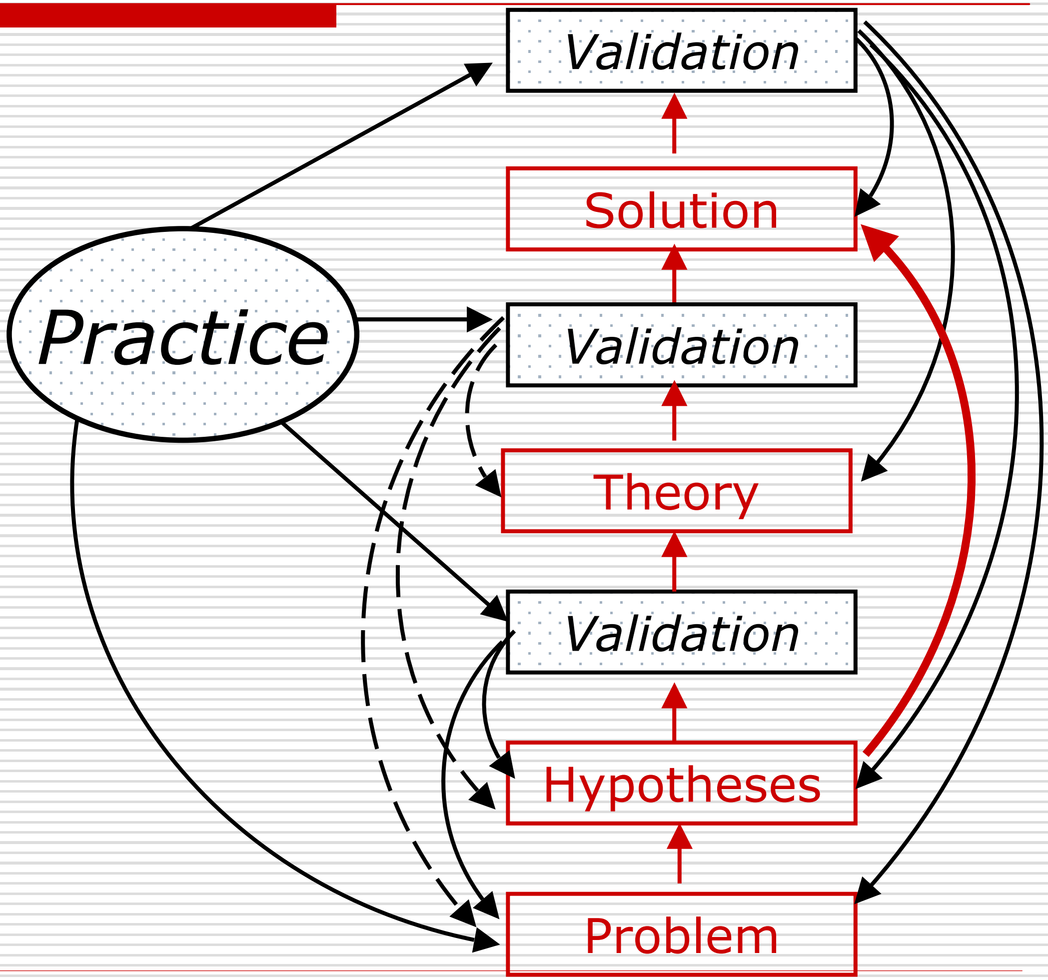
Risk valuation and ratings



Many probability distributions are not adequate enough in the domain of low probabilities

The adequacy of scientific methods. Validation criteria

- From the problem to a solution through hypothesis and theory
- Validation. Practice as one of the main verification tools



Areas of application and adequacy of mathematical models

- Conventional financial instruments (mortgages, annuities)
- Evaluation of investment portfolios (stocks, bonds)
- Investment strategies and trading
- Risk valuation
- Composite high leverage financial instruments (CDO)
- Monetary policy (emission)

Summary of some issues to be addressed by financial mathematics

- Encourage development of *adequate* and *practical* mathematical methods. Complexity, in general, *does not improve* the adequacy. (Compiling a list of pertinent issues?)
- Educate public and governing institutions of the inherent risk associated with the high leverage financial instruments and their potential ability to induce the market's instability
- Risk valuation methods require more objectivity and maybe revision of some fundamentals
- Stability of financial markets and their impact to the overall economy should become an important direction of research within the area of financial mathematics